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Asymptotic quantization for solution manifolds of some infinite dimensional Hamiltonian systems

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Abstract

The solution manifolds of some classes of Hamiltonian systems in Hilbert phase spaces are considered. Pseudodifferential operators with symbols on these manifolds are defined.

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Introduction

In this work we define an algebra of pseudodifferential operators (PDO) with symbols on infinite-dimensional symplectic manifolds of solutions of some infinite-dimensional Hamilton equations.

In the finite-dimensional case the theory of PDO with symbols on symplectic manifolds (or asymptotic quantization theory) was considered in [KM1,KM2]. In these works PDO were constructed using asymptotic solutions of Schrödinger equations corresponding to the transition mappings of symplectic manifolds. These solutions are obtained by Maslov canonical operator method, see e.g. [M,MF,KM2].

Differential and pseudodifferential operators with infinite-dimensional linear phase spaces have been studied e.g. in [AKR,AR,BK,B,BV,Kh,SU,S]. The development of asymptotic

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methods in infinite-dimensional analysis began in the works [AHK2,AHK3,ABHK,Re] (stationary phase method for infinite-dimensional oscillatory integrals, see [AB] for further developments and applications). The results of these authors were applied to the investigation of some classes of infinite-dimensional PDO in [D1,D2,D3,D5] in which an infinite-dimensional version of Maslov canonical operator method was given.

In the present work we consider two types of solution manifolds for Hamilton equations in a rigged Hilbert phase space $\mathcal{H}_+^2 \subset \mathcal{H}^2 \subset \mathcal{H}_-^2$. The corresponding Hamilton function H is supposed to have the form $H = B + h$, where B is a second-order polynomial continuous on $\mathcal{H}_-^2$, and h is the Fourier transform of a complex measure on $\mathcal{H}^2$. The particular form of the corresponding transition mappings allows to apply the asymptotic methods mentioned above and to construct PDO with symbols on these manifolds (at least mod $O(\hbar)$). The PDO with symbols on local maps used in this construction are defined as functions of self-adjoint operators satisfying canonical commutation relations in $L_2(\mathcal{H}_-, \eta)$, where η is an $\mathcal{H}_+$-quasi invariant Gaussian measure.

We pay attention here only to analytical questions considering the case of simply connected manifolds. The topological meaning of cohomological conditions for the existence of global PDO associated with homologically nontrivial manifolds ("quantization conditions") which can be calculated (at least heuristically) as in the finite-dimensional asymptotic quantization theory is however not clear yet.

Let us remark that our considerations have some connection with the approach initiated in [Se]. We would like to mention also the paper [AP] where the geometric quantization method for some infinite dimensional manifolds has been considered.

The content of the present paper is as follows. In Section 1 we consider some necessary technical questions. In Section 2 we introduce PDO with symbols on Hilbert phase space and obtain the main formulae of symbolic calculus, namely the formulae giving the composition and the commutator of two PDOs. In Section 3 the solutions of the corresponding Schrödinger equations are constructed in the form of PDO with oscillatory symbols. In Section 4 we discuss the geometry of the symplectic manifolds mentioned above and give an invariant definition of PDO with symbols on them.

1. Some questions of analysis is the spaces of Fourier transforms

Let $\mathcal{B}$ be a real Banach space and $\mathcal{P}$ be a complex one. Consider the space $\mathcal{M}(\mathcal{B}, \mathcal{P})$ of $\mathcal{P}$-valued measures μ on the dual space $\mathcal{B}'$ with bounded variation $\operatorname{Var}|\mu|$, where $|\mu| = \sup_{\xi \in \mathcal{P}', \|\xi\| \le 1} |\langle \xi, \mu \rangle|$ is the variation of the positive measure $|\langle \xi, \mu \rangle|$. Denote by $M_\lambda(\mathcal{B}, \mathcal{P})$ the subspace of measures satisfying the condition

$$\int e^{\lambda \|p\|} |\mu|(dp) < \infty, \quad \lambda \in \mathbb{R}^1. \tag{1}$$

For any λ the space M_λ is a Banach space with the norm $\|\mu\|_\lambda = \int e^{\lambda\|p\|}|\mu|(dp)$ ($M_0 = M$). We set $M_\infty = \cap M_\lambda$, $M_{+0} = \cup M_\lambda$.

Let $\mathcal{A}$ be another Banach space, and let $\nu \in M_\infty(\mathcal{A}\times\mathcal{B},\mathcal{P})$, $\mu \in M_\lambda(\mathcal{A},\mathcal{B})$. Consider the Fourier transforms f and φ of ν and μ respectively, i.e. $f(x,y) = \int e^{i\langle q,x\rangle+i\langle p,y\rangle}\nu(dq,dp)$, $\varphi(x) = \int e^{i\langle q,x\rangle}\mu(dq)$, $x \in \mathcal{A}$, $y \in \mathcal{B}$. Let $A : \mathcal{A} \to \mathcal{B}$ be a bounded linear operator and set $\psi \equiv A + \varphi$.

Theorem 1. *The composition $f(\cdot,\psi(\cdot))$ is the Fourier transform of some measure $\beta \in M_\lambda(\mathcal{A},\mathcal{B})$. If the measure ν is differentiable in the direction h, β is also differentiable in this direction.*

Proof. We first remark that because of (1) f is an entire mapping [AHK2]. Thus we have

$$f(x,y+z) = f(x,y) + \sum_{k=1}^{\infty}\frac{1}{k!}f_k(x,y)(z), \tag{2}$$

where $f_k(x,y)$ is the k-linear $\mathcal{P}$-valued functional,

$$f_k(x,y)(z) = \int e^{i\langle q,x\rangle+i\langle p,y\rangle}\langle p,z\rangle^k\,\nu(dq,dp). \tag{3}$$

Let now $y = Ax$, $z = \varphi(x)$. Then:

$$\begin{aligned} f_k(x,Ax)(\varphi(x)) &= \int e^{i\langle q,x\rangle+i\langle A^*p,x\rangle}\left\langle p, \int e^{i\langle\theta,x\rangle}\mu(d\theta)\right\rangle^k \nu(dq,dp) \\ &= \int e^{i\langle q+A^*p+\theta,x\rangle}\langle p,\ \mu\rangle^{*k}(d\theta)\nu(dq,dp), \end{aligned} \tag{4}$$

where the symbol ^{*k} means k-multiple convolution and A^* is the adjoint of A. Therefore

$$f(x,Ax+\varphi(x)) = \int e^{i\langle q+A^*p+\theta,x\rangle}e^{\langle p,\mu\rangle}(d\theta)\nu(dq,dp) \tag{5}$$

(the exponent $e^{\langle p,\mu\rangle} \equiv \sum_{k=0}^{\infty}(1/k!)\langle p,\mu\rangle^{*k}$ exists as an element of the Banach algebra $M_\lambda(\mathcal{A},\mathcal{B})$).

Let β be the image of the measure $e^{\langle p,\mu\rangle}(d\theta)\nu(dq,dp)$ under the mapping $(q,p,\theta) \mapsto q + A^*p + \theta$.

It is easy to see that $\|\beta\|_\lambda \le c\|\nu\|_{\lambda+\|\mu\|_\lambda}$ and that the differentiability of ν implies the differentiability of β. □

Consider now a real separable Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$ rigged by Hilbert spaces $\mathcal{H}_+,\mathcal{H}_-$ with Hilbert–Schmidt embeddings:

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-. \tag{6}$$

Let $\mathcal{F}(\mathcal{H})$ be the space of mappings $f:\mathcal{H} \to \mathcal{H}$ which are Fourier transforms of measures of the class $M_\infty(\mathcal{H},\mathcal{H}_\mathbb{C})(\mathcal{H}_\mathbb{C}$ is the complexification of $\mathcal{H})$. $\mathcal{F}(\mathcal{H})$ is a subspace of the Banach space $\mathcal{F}_\lambda(\mathcal{H})$ of $\mathcal{H}$-valued Fourier transforms of the measures of the class $M_\lambda(\mathcal{H},\mathcal{H}_\mathbb{C})$.

Let $\mathcal{K}(\mathcal{H})$ be the Banach space of bounded linear operators $B : \mathcal{H}_- \to \mathcal{H}_+$ with trace-norm. Consider the space $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \times \mathcal{F}(\mathcal{H}) \times \mathcal{H}$ of the mappings $\mathcal{H} \ni x \mapsto Bx + l(x) + h \in \mathcal{H}, B \in \mathcal{K}(\mathcal{H}), l \in \mathcal{F}(\mathcal{H}), h \in \mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a subspace of the Banach space $\mathcal{B}_\lambda(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \times \mathcal{F}_\lambda(\mathcal{H}) \times \mathcal{H}$.

Let us consider the differential equation in $\mathcal{H}$:

$$\dot{x}(t) = L(x(t), t), \qquad x(0) = y, \quad t \in [0, T] \subset \mathbb{R}^1. \tag{7}$$

Assume that $L(\cdot, t) \in \mathcal{B}(\mathcal{H})$ and the mapping $[0, T] \ni t \mapsto L(\cdot, t) \in C^1(\mathcal{H}_-, \mathcal{H})$ is differentiable.

Theorem 2. *A solution of the Cauchy problem* (7) *exists for any initial data y and the mapping $\gamma(t) : y \mapsto x(t)$ has the form* id + $\varphi(t)$ *with $\varphi(t) \in \mathcal{B}(\mathcal{H})$. If the measure corresponding to L is differentiable in some direction $h \in \mathcal{H}_+$, then the same property also holds for the measure corresponding to φ.*

Proof. The existence of global solutions follows from the results of [H]. We remark that (7) is equivalent to the equation

$$\dot{\varphi}(t) = \mathcal{L}(\mathrm{id} + \varphi(t), t), \qquad \varphi(0) = 0 \tag{8}$$

in the space of mappings $Y : \mathcal{H} \to \mathcal{H}$, where $\mathcal{L}(Y, t)(x) = L(Y(x), t)$. It is easy to check that the mapping $\mathcal{L}$ is differentiable in any space $\mathcal{B}_\lambda(\mathcal{H})$, which is enough for the solvability of (8) in $\mathcal{B}_\lambda(\mathcal{H})$. Notice at least that by Theorem 1 the measure corresponding to $\mathcal{L}(g, t)$ is h-smooth for any $g \in \mathcal{B}_\lambda$, if the same property has the measure corresponding to $L(t)$. □

Let $\mathcal{F}(\mathcal{H}, \mathbb{R}^1)$ be the space of real-valued functions on $\mathcal{H}$ which are Fourier transforms of measures from $M_\infty(\mathcal{H}, \mathbb{C}^1)$. $\mathcal{F}(\mathcal{H}, \mathbb{R}^1)$ is a subspace of the Banach space $\mathcal{F}_\lambda(\mathcal{H}, \mathbb{R}^1)$.

We will need the following fact.

Lemma 1. *Let $\gamma \in \mathcal{F}(\mathcal{H}), \gamma(x) = \int \mathrm{e}^{\mathrm{i}(p,x)} \nu(\mathrm{d}x)$ for some $\mathcal{H}_\mathbb{C}$-valued measure ν and $\gamma = \mathrm{d}S$ for some function $S : \mathcal{H} \to \mathbb{R}^1$. Suppose that ν is smooth in some two directions $h_1, h_2 \in \mathcal{H}_+$. Then $S \in \mathcal{F}(\mathcal{H}, \mathbb{R}^1)$ (modulo an additive constant).*

Proof. For any $y \in \mathcal{H}$ set $y_1 = \langle y, h_1 \rangle, y_2 = \langle y, h_2 \rangle$. Obviously we have $\partial \gamma_2(x)/\partial x_1 = \partial \gamma_1(x)/\partial x_2$, i.e. $p_1 \nu_2(\mathrm{d}p) = p_2 \nu_1(\mathrm{d}p)$. Consider the measure $\mu = (1/p_1)\nu_1 = (1/p_2)\nu_2$. Let us prove that $\mu \in M_\infty(\mathcal{H}, \mathcal{H}_\mathbb{C})$. Consider the projection θ of ν into the plane $\{h_1, h_2\}$. Because of h_1, h_2-smoothness of ν we have $\theta(\mathrm{d}p_1, \mathrm{d}p_2) = f(p_1, p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$ for some smooth function f. Evidently,

$$\begin{aligned} &\int \mathrm{e}^{\lambda \|p\|} |\mu(\mathrm{d}p)| \\ &\quad \le \int \exp\left[\lambda \|p\| \frac{|p_1| + |p_2|}{\sqrt{p_1^2 + p_2^2}}\right] |\mu(\mathrm{d}p)| \end{aligned}$$

$$\leq c \int \exp\left[\lambda \|(p_1, p_2)\| \frac{|f_1(p_1, p_2| + |f_2(p_1, p_2)|}{\sqrt{p_1^2 + p_2^2}}\right] \mathrm{d}p_1 \, \mathrm{d}p_2 < \infty. \qquad \square$$

2. Pseudodifferential operators with symbols on a Hilbert phase space

Introduce now the Hilbert phase space $\mathcal{H}^2 = \mathcal{H}_x \times \mathcal{H}_p, \mathcal{H}_x = \mathcal{H}_p = \mathcal{H},$ with the natural symplectic form $\mathrm{d}p \wedge \mathrm{d}x$ generated by the scalar product $\langle \cdot, \cdot \rangle$. $\mathcal{H}^2$ has the natural rigging

$$\mathcal{H}_+^2 \subset \mathcal{H}^2 \subset \mathcal{H}_-^2 \tag{9}$$

with $\mathcal{H}_+^2 = \mathcal{H}_+ \times \mathcal{H}_+, \mathcal{H}_-^2 = \mathcal{H}_- \times \mathcal{H}_-$.

Consider a Gaussian measure η on $\mathcal{H}_-$ defined by the characteristic functional

$$\chi_\eta(t) = \int \mathrm{e}^{\mathrm{i}\langle t, x \rangle} \eta(\mathrm{d}x) = \mathrm{e}^{-\langle Bt, t \rangle/2}, \quad t \in \mathcal{H}, \tag{10}$$

where B is a bounded invertible symmetric positive operator in $\mathcal{H}$. It is well known that η is σ-additive in $\mathcal{H}_-$ and quasi-invariant with respect to $\mathcal{H}$: $\eta(\mathrm{d}x + t) = \rho_t(x)\eta(\mathrm{d}x), t \in \mathcal{H}$, where

$$\rho_t(x) = \mathrm{e}^{-\langle Bt, t \rangle/2 - \langle Bt, x \rangle}. \tag{11}$$

Let C be a symmetric bounded operator in $\mathcal{H}$. Consider the function

$$\alpha_t(x) = \mathrm{e}^{\mathrm{i}\langle Ct, t \rangle/2 + \mathrm{i}\langle Ct, x \rangle}, \quad t \in \mathcal{H}. \tag{12}$$

$\alpha_t(\cdot)$ is measurable with respect to η because of the measurability of the linear functional $\langle Ct, \cdot \rangle$. It satisfies the equation

$$\alpha_{t+\tau}(x) = \alpha_t(x)\alpha_\tau(x + t) \tag{13}$$

and will be called a Gaussian cocycle.

Consider the family U of commuting unitary operators $U_t, t \in \mathcal{H}_+$, in $L_2(\mathcal{H}_-, \eta)$ defined by the formula

$$U_t\varphi(x) = \alpha_t(x)\sqrt{\rho_t(x)}\varphi(x + t). \tag{14}$$

The following lemma is proven in [Sa,D4].

Lemma 2. *The family U_t is strongly continuous in t and cyclic with the cyclic vector $\varphi = 1$ (i.e.* $\mathrm{span}_{t \in \mathcal{H}}\{U_t 1\}$ *is dense in $L_2(\mathcal{H}_-, \eta)$). The family U_t is unitary equivalent to the family of operators V_t of multiplication by functions* $\mathrm{e}^{\mathrm{i}\langle t, \cdot \rangle}$ *in the space $L_2(\mathcal{H}_-, \nu)$, where ν is Gaussian measure on $\mathcal{H}_-$ with characteristic functional*

$$\chi_\nu(t) = \int \mathrm{e}^{\mathrm{i}\langle t, x \rangle} \nu(\mathrm{d}x) = \mathrm{e}^{-\langle Dt, t \rangle/2}, \quad t \in \mathcal{H}, \tag{15}$$

$$D = B^{-1} + CBC. \qquad \square$$

Let $F : L_2(\mathcal{H}_-, \eta) \mapsto L_2(\mathcal{H}_-, \nu)$ be the corresponding intertwining operator: $FU_t = V_t F$.

Remark 1.
(1) If η is the standard Gaussian measure associated with $\mathcal{H}$ and $\alpha \equiv 1$, the operator F coincides with the well-known Fourier–Wiener transform, see e.g. [BK,DF].
(2) The spectral measure ν depends on the choice of a particular cyclic vector for U. Different cyclic vectors give equivalent measures and therefore unitary equivalent intertwining operators.
(3) The operators U_t and the operators W_τ of multiplication by fuctions $e^{i\langle\cdot,\tau\rangle}$, $\tau \in \mathcal{H}$, satisfy the canonical commutation relations in the Weyl form

$$U_t W_\tau = e^{i\langle t,\tau\rangle} W_\tau U_t. \tag{16}$$

(4) The analogue of Lemma 2 holds true also in the case of unbounded operators B and C satisfying some additional conditions [D4].

Let us consider now functions of the generators x and D of the families W and U. Let H be a function (symbol) on $\mathcal{H}_-^2$. Define formally a pseudodifferential operator (PDO) $\hat{H} \equiv \hat{H}_{\eta,\alpha}$ in $L_2(\mathcal{H}_-, \eta)$ using the operator F:

$$\hat{H}\varphi(x) \equiv H(x, i\hbar D)\varphi(x) = F^{-1}_{p\to x} F_{y\to p}\left[H\left(\tfrac{1}{2}(x+y), \hbar p\right)\varphi(y)\right] \tag{17}$$

for any $\hbar \in (0, 1]$. The sign "$p \to x$" means that the corresponding operator is applied to a function of p and the result is considered as the function of x.

Further we will specify the definition of PDO for particular classes of symbols.

Remark 2. Originally infinite-dimensional differential operators (with constant coefficients) were considered as functions of x and D in [SU]. The spectral transformation F was applied there to their investigation.

Properties 1.
(1) PDO with real symbols are symmetric operators.
(2) PDO defined by different measures and cocycles are unitary equivalent iff the corresponding measures and cocycles are equivalent.
(3) In the case of $\mathcal{H} = \mathbb{R}^n$, $\alpha_t(x) = f(x+t)/f(x)$, $\rho_t(x) = g(x+t)/g(x)$, where $f(x) = e^{-\langle Bx,x\rangle/2}$, $g(x) = e^{i\langle Cx,x\rangle/2}$ for some constant matrices B, C. Therefore

$$\hat{H}_{\eta,\alpha} = r(x)^{-1} H(x, i\hbar\partial/\partial x) r(x), \tag{18}$$

where $r = f\sqrt{g}$ and $H(x, i\hbar\partial/\partial x)$ is the PDO defined by the usual Fourier transform.
(4) Let H be a continuous polynomial on $\mathcal{H}_-^2$, i.e. $H = \sum_{n=0}^m B_n$, where B_n is an n-linear continuous functional on $\mathcal{H}_-^2$. Then $\hat{H}$ is the differential operator with polynomial coefficients with natural invariant domain of definition $\mathbb{D}$ which consists of $\mathcal{H}_+$-smooth functions f on $\mathcal{H}_-$ satisfying the condition

$$\|d^n f(x)\|_n \|x\|_-^m \in L_2(\mathcal{H}_-^2, \eta) \tag{19}$$

for any n, m, where $\|\cdot\|_n$ is the norm in n-tensor power $\mathcal{H}_-^{\otimes n}$ of $\mathcal{H}_-$.

Consider a measure $\mu \in M_\infty(\mathcal{H}, \mathbb{C})$ and $B \in \mathcal{H}_+^{\otimes n+m}$. Define a linear functional ξ on the space $\mathbb{D}$ in the following way:

$$\int f(y)\xi(\mathrm{d}y) = \int \langle B, \mathrm{d}^n f(y) \otimes y^{\otimes m}\rangle \mu(\mathrm{d}y) \tag{20}$$

(we use the notation $\int f(y)\xi(\mathrm{d}y) = \xi(f)$).

The space of such functionals will be denoted by $MP(\mathcal{H})$.

Consider the space $C_{\mathrm{pol}}(\mathcal{H}_-, L_2(\mathcal{H}_-, \eta))$ of $\mathcal{H}_+$-smooth mappings $\varphi : \mathcal{H}_- \to L_2(\mathcal{H}_-, \eta)$ satisfying the condition

$$\|\mathrm{d}\varphi\|_n \le c(1 + \|x\|_-^m) \tag{21}$$

for some constants $c = c(n)$, $m = m(n)$ and any n (where $\|\cdot\|_n$ is the norm in the space $\mathcal{H}_-^{\otimes n} \otimes L_2(\mathcal{H}_-, \eta)$). In other words, φ is a function on $\mathcal{H}_- \times \mathcal{H}_-$ and $\varphi(\cdot, y) \in L_2(\mathcal{H}_-, \eta)$ for any $y \in \mathcal{H}_-$. We need the following technical fact.

Lemma 3. *For $\varphi \in C_{\mathrm{pol}}(\mathcal{H}_-, L_2(\mathcal{H}, \eta))$ the function $\Phi(\cdot) = \int \varphi(\cdot, y)\xi(\mathrm{d}y) \in L_2(\mathcal{H}_-, \eta)$ and*

$$F_{p\to x}\Phi(p) = \int F_{p\to x}\varphi(p, y)\xi(\mathrm{d}y). \tag{22}$$

Proof. For $f \in L_2(\mathcal{H}_-, \eta)$

$$\begin{aligned} &\left|\int \Phi(x) f(x)\eta(\mathrm{d}x)\right| \\ &\quad = \left|\int \left(\int \langle B, \mathrm{d}_y^n\varphi(x, y) \otimes y^{\otimes m}\rangle \mu(\mathrm{d}y)\right) f(x)\eta(\mathrm{d}x)\right| \\ &\quad = \left|\int \left(\int \langle B, \mathrm{d}_y^n\varphi(x, y) \otimes y^{\otimes m}\rangle f(x)\eta(\mathrm{d}x)\right) \mu(\mathrm{d}y)\right| \\ &\quad \le C\|f\|^2_{L_2(\mathcal{H}_-, \eta)}\|B\|_+^2 \operatorname{Var}|\mu| \end{aligned}$$

because of (21), and

$$\int \Phi(x) f(x)\eta(\mathrm{d}x) = \int \left(\int \varphi(x, y) f(x)\eta(\mathrm{d}x)\right) \xi(\mathrm{d}y).$$

Then for $g \in L_2(\mathcal{H}_-, \nu)$

$$\begin{aligned} \int (F\Phi)(x)g(x)\nu(\mathrm{d}x) &= \int \Phi(x)(F^{-1}g)(x)\eta(\mathrm{d}x) \\ &= \int \left(\int \varphi(x, y)(F^{-1}g)(x)\eta(\mathrm{d}x)\right) \xi(\mathrm{d}y) \\ &= \int F_{p\to x}\varphi(p, y)\xi(\mathrm{d}y). \end{aligned}$$

□

Let us introduce now the following spaces of functions on $\mathcal{H}^2$.

$\mathcal{F}$—the space of functions which are Fourier transforms of measures from $M_\infty(\mathcal{H}^2, \mathbb{C})$.
$\mathcal{P}^n$—the space of $\mathcal{H}^2_-$-smooth polynomials of the order $\le n$; $\mathbb{P} = \cup \mathbb{P}^n$.
$\mathcal{D}^n = \mathcal{P}^n + \mathcal{F}$(i.e. $\mathcal{D} \ni f = P + \varphi, \mathcal{P} \in \mathcal{P}^n, \varphi \in \mathcal{F}$).
$\mathcal{E}$—the space of Fourier transforms of functionals of the class $MP(\mathcal{H}^2)$. Obviously

$$\mathcal{E} = \{H\colon H(x,p) = \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle} P(x,p,x',p')\theta(dx',dp'),$$
$$P \in \mathcal{P}(\mathcal{H}^4_-), \theta \in M_\infty\}$$

Proposition 1. *For $H \in \mathcal{F}$ the PDO $\hat{H}$ is bounded operator in $L_2(\mathcal{H}_-, \eta)$ and leaves invariant the space $\mathbb{D}$.*

Proof. Let $H(x,p) = \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle}\theta(dx',dp')$, $f \in L_2(\mathcal{H}_-, \eta)$ Then:

$$\begin{aligned}\hat{H}f(x) &= \int e^{i\langle x,x'\rangle/2} F^{-1}_{p\to x}[e^{i\hbar\langle p,p'\rangle} F_{y\to p}[e^{i\langle y,x'\rangle/2} f(y)]]\theta(dx',dp') \\ &= \int e^{i\langle x,x'\rangle/2} U_{\hbar p'}[e^{i\langle x,x'\rangle/2} f(x)]\theta(dx',dp') \\ &= \int e^{i\langle x,x'\rangle} U_{\hbar p'} f(x) e^{i\hbar\langle x',p'\rangle/2}\theta(dx',dp') \end{aligned} \tag{23}$$

and $\hat{H}f \in L_2(\mathcal{H}_-, \eta)$ by Lemma 3. It is easy to check that for $h_n \in \mathcal{H}_+^{\otimes n}$, $h \in \mathcal{H}_+$ and $f \in \mathbb{D}$

$$\langle d^n \hat{H} f(x), h_n\rangle \langle x, h\rangle^m \in L_2(H_-, \eta) \tag{24}$$

which is equivalent to (19). □

Proposition 2. *For $H \in \mathcal{E}$ the operator $\hat{H}$ is defined on $\mathbb{D}$ and leaves $\mathbb{D}$ invariant.*

Proof. The formula (22) shows that (23) holds for $f \in \mathbb{D}$ and $\theta \in MP(\mathcal{H}^2)$. Now the proof can be completed similarly as the proof of Proposition 1. □

Proposition 3. *Let $H, G \in \mathcal{E}(\mathcal{H}^2)$, $H(x,p) = \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle}\theta(dx',dp')$ with $\theta \in MP(\mathcal{H}^2)$. Then, $\hat{H}\hat{G} = \hat{N}_1$, $\hat{G}\hat{H} = \hat{N}_2$, where $N_1, N_2 \in \mathcal{E}$ and*

$$N_1(x,p) = \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle} G(x + \tfrac{1}{2}\hbar p', p - \tfrac{1}{2}\hbar x')\theta(dx',dp'), \tag{25}$$

$$N_2(x,p) = \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle} G(x - \tfrac{1}{2}\hbar p', p + \tfrac{1}{2}\hbar x')\theta(dx',dp'). \tag{26}$$

Proof. Let $G(x,p) = \int e^{i\langle x,x''\rangle + i\langle p,p''\rangle}\theta_G(dx'',dp'')$. Then:

$$\begin{aligned} N_1(x,p) &= \int e^{i\langle x,x'\rangle + i\langle p,p'\rangle} \int e^{i\langle x+\hbar p'/2, x''\rangle + i\langle p-\hbar x'/2, p''\rangle}\theta_G(dx'',dp'')\theta(dx'',dp'') \\ &= \int e^{i\langle x,x'+x''\rangle + i\langle p,p'+p''\rangle} e^{i\hbar(\langle p',x''\rangle - \langle x',p''\rangle)/2}\theta_G(dx'',dp'')\theta(dx',dp') \end{aligned}$$

$$= \int e^{i\langle x,y\rangle + i\langle p,q\rangle} \theta_{N_1}(dy, dq),$$

where θ_{N_1} is the image of the functional $e^{i\hbar(\langle p',x''\rangle - \langle x',p''\rangle)/2}\theta_G(dx'', dp'')\theta(dx', dp')$ under the mapping $x' + x'' \mapsto y$, $p' + p'' \mapsto q$. It is easy to see that $\theta_{N_1} \in MP(\mathcal{H}^2)$.

Let us calculate now the symbol of the composition $\hat{G}\hat{H}$. By formula (23),

$$\begin{aligned}
\hat{H}\hat{G}f(x) &= \int e^{i\langle x,x'\rangle} U_{\hbar p'}(\hat{G}f)(x) e^{i\hbar\langle x',p'\rangle} \Theta(dx', dp') \\
&= \int e^{i\langle x,x'\rangle} e^{i\langle (x+\hbar p'),x''\rangle} U_{\hbar(p'+p'')} f(x) \\
&\quad \times e^{i\hbar(\langle x',p'\rangle + \langle x'',p''\rangle)/2} \Theta_G(dx'', dp'')\Theta(dx', dp') \\
&= \int e^{i\langle x,x'+x''\rangle} U_{\hbar(p'+p'')} f(x) e^{i\hbar\langle x'+x'',p'+p''\rangle/2} \\
&\quad \times e^{-i\hbar(\langle x',p''\rangle - \langle x'',p'\rangle)/2} \Theta_G(dx'', dp'')\Theta(dx', dp')
\end{aligned}$$

The symbol N_2 can be calculated similarly. □

Corollary 1 (Commutation formula). *Let $G, H \in \mathcal{E}(\mathcal{H}^2)$. Then,*

$$(i/\hbar)[\hat{H}, \hat{G}] = \widehat{\{H, G\}} + O(\hbar^2), \tag{27}$$

where the remainder is estimated in the strong sense on $\mathbb{D}$ ($\{\cdot,\cdot\}$ means the standard Poisson bracket on $\mathcal{H}^2$).

Proof. It is enough to consider two terms of the decomposition of the right-hand sides of (25) and (26) expanded in power series in $\hbar$. □

We shall now discuss PDO with symbols of another type. We first introduce unitary operators T_K resp. $\tilde{T}_K$ in $L_2(\mathcal{H}_-, \eta)$ resp. $L_2(\mathcal{H}_-, \nu)$, $K \in \mathcal{K}(\mathcal{H})$:

$$T_K f(x) = \sqrt{s(x)} f((I + K)x), \tilde{T}_K \varphi(p) = \sqrt{\tilde{s}(p)} \varphi(p),$$

where $s_K(x) = \eta((I + K)\,dx)/\eta(dx)$, $\tilde{s}_K(p) = \nu((I + K)\,dp)/\nu(dp)$.

Proposition 4. *Let $H(x, p) = e^{iP(x,p)/\hbar}$, where $P \in \mathcal{P}^2(\mathcal{H}^2_-)$. Then,*

$$\begin{aligned}
\hat{H}f(x) &= (\det(I + K_1)(I + K_2))^{-1} \\
&\quad \times e^{iP_1(x)} T_{K_1}(F^{-1}_{p\to x} e^{iP_2(p)} \tilde{T}_{K_2}(F_{y\to p} e^{iP_3(y)} f(y))),
\end{aligned} \tag{28}$$

where $P_1, P_2, P_3 \in \mathcal{P}^2(\mathcal{H}_-)$, $K_1, K_2 \in \mathcal{K}(\mathcal{H})$ and depend on $\hbar$.

Proof. Let us prove the statement in the case $H(x, p) = e^{2i\langle Kx,p\rangle/\hbar}$. By definition,

$$\hat{H}f(x) = F^{-1}_{p\to x} e^{i\langle Kx,p\rangle/2} F_{y\to p}(e^{i\langle Ky,p\rangle} f(y)).$$

Denote by $\tilde{U}_t$ the operator in $L_2(\mathcal{H}_-, \nu)$ defined in the following way:

$$\tilde{U}_t f(p) = F_{y\to p} e^{i\langle y,t\rangle} F^{-1}_{p\to y}.$$

The family $\tilde{U}_t, t \in \mathcal{H}$ is cyclic and satisfies canonical commutation relations with the family W_t. Therefore $\tilde{U}_t f(p) = \beta_t(p)\sqrt{\tilde{\rho}_t(p)} f(p+t)$, where $\tilde{\rho}_t(p) = \nu(\mathrm{d}p + t)/\nu(\mathrm{d}p)$ and β is some cocycle. This cocyle is Gaussian, $\beta_t(x) = \mathrm{e}^{\mathrm{i}\langle At, t\rangle/2 + \mathrm{i}\langle At, x\rangle}$. The explicit expression for the operator A is given in [D4].

It is easy to see that $\rho_{-K_x}(x) = s_K \det(I-K)^{-1}, \tilde{\rho}_{Kp}(p) = \tilde{s}_K \det(I+K)^{-1}$.

Then,

$$\hat{H} f(x) = (\det(I-K)(I+K))^{-1} a(x) T_{-K}(F^{-1}_{p\to x} b(p) \tilde{T}_K (F_{y\to p} f(y))),$$

where $a(x) = \alpha_{Kx}(x), b(p) = \beta_{Kp}(p)$.

The general case can be proved similarly. □

Corollary 2. *The PDO with symbol of the form*

$$H(x,p) = \int \mathrm{e}^{\mathrm{i}\langle x, x'\rangle + \mathrm{i}\langle p, p'\rangle} \mathrm{e}^{\mathrm{i}P(x,p,x',p')/\hbar} \theta(\mathrm{d}x', \mathrm{d}p'), \tag{29}$$

$P \in \mathcal{P}^2(\mathcal{H}^4_-), \theta \in \mathcal{E}$, *is correctly defined on* $\mathbb{D}$ *and leaves* $\mathbb{D}$ *invariant.*

Corollary 3. *For H of the form* (29) *and* $G \in \mathcal{E}$ *formulae* (26) *and* (27) *hold true.*

3. PDO with oscillatory symbols and solutions of Schrödinger equations

We now want to consider PDO with symbols depending on the parameter $\hbar$ and oscillating in 0 and to obtain with their aid the solutions of Schrödinger equations for PDO with $\mathcal{D}^2$-symbols.

For this we need PDO of another type. If $H \in \mathcal{D}^2$, the operator $H(x, \mathrm{i}\hbar\partial/\partial x)$ can be defined [S,Kh]:

$$H(x, \mathrm{i}\hbar\partial/\partial x)\varphi(x) = \int^{\sim} \mathrm{e}^{-\mathrm{i}\langle p, x-y\rangle/\hbar} H(x,p)\varphi(y)\,\mathrm{d}p\,\mathrm{d}y. \tag{30}$$

Here $\int^{\sim}$ is a normalized integral defined by Parseval formula

$$\int^{\sim} \mathrm{e}^{\mathrm{i}\langle Ax, x\rangle/\hbar} f(x)\,\mathrm{d}x = \int \mathrm{e}^{-\langle A^{-1}y, y\rangle/\hbar} \mu_f(\mathrm{d}y) \tag{31}$$

for f an $\hbar$-Fourier transform of a measure or distribution μ_f: $f(x) = \int \mathrm{e}^{\mathrm{i}\langle p, x\rangle/\hbar} \mu_f(\mathrm{d}p)$ (see [AHK1,S,Kh]).

We can now rewrite the composition formula (25) in the following form:

$$\hat{H}\hat{G} = \hat{N}, \qquad N(x,p) = H\left(x + \tfrac{1}{2}\mathrm{i}\hbar\partial/\partial p,\ p - \tfrac{1}{2}\mathrm{i}\hbar\partial/\partial x\right) G(x,p). \tag{32}$$

We assume from now that the measures corresponding to the elements of the spaces $\mathcal{D}^n$ resp. $\mathcal{B}$ satisfy the conditions of Lemma 1, i.e. are smooth in some two directions $h_1, h_2 \in \mathcal{H}_+$ resp. $\mathcal{H}^2_+$.

Let us consider the (mod $\hbar$) Schrödinger equation

$$\left[\frac{\partial}{\partial t} - \frac{\mathrm{i}}{\hbar}\hat{H}_t\right] T_t = \mathrm{O}(\hbar), \tag{33}$$
$$T_0 = \mathrm{id},$$

where $H = H_t \in \mathcal{D}^2$ and is real valued. Assume that the mapping $t \to H(t) \in \mathcal{B}^2_\lambda(\mathcal{H}^2_-, \mathbb{R}^1)$ is 2-differentiable for any λ.

Consider first the classical Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{34}$$

in $\mathcal{H}^2$.

Denote by $\gamma_H(t,\tau), t, \tau \in [0,T]$ the corresponding shift along integral curves of (34).

Let $\mathrm{Symp}_{\mathcal{F}}(\mathcal{H}^2)$ be the space of symplectomorphisms γ of $\mathcal{H}^2$ such that $\gamma - \mathrm{id} \in \mathcal{B}(\mathcal{H}^2)$. It follows from Theorem 1 that $\mathrm{Symp}_{\mathcal{F}}(\mathcal{H}^2)$ is a group.

Proposition 4.
(1) *For any* $t, \tau \in [0,T], \gamma_H(t,\tau) \in \mathrm{Symp}_{\mathcal{F}}(\mathcal{H}^2)$.
(2) *For any* $\gamma \in \mathrm{Symp}_{\mathcal{F}}(\mathcal{H}^2)$ *there exists a Hamiltonian* H *of the above type such that* $\gamma = \gamma_H\,(0,1)$.

Proof. The first statement follows immediately from Theorem 2. The construction of a Hamilton vector field corresponding to γ is given in the finite-dimensional case in [KN]. It is easy to check that this construction remains true in the case of a Hilbert phase space and that this vector field belongs to $\mathcal{B}(\mathcal{H}^2)$. Lemma 1 implies that the corresponding Hamiltonian belongs to $\mathcal{D}^2$. □

Theorem 3. *The* (mod $\hbar$) *Schrödinger equation* (33) *has a solution of the form* $T_t = \hat{U}_t$, *where* $U_t(x,p) = \mathrm{e}^{\mathrm{i}P_t(x,p)/\hbar} + S_{\hbar,t}(x,p)$, $P \in \mathcal{P}^2$, $S \in \mathcal{F}$. *The remainder is bounded in the strong sense on* $\mathbb{D}$.

Proof. An equation for the symbol U can be obtained by the formula (26):

$$\left[(\partial/\partial t) - (\mathrm{i}/\hbar)H_t\left(x + \tfrac{1}{2}\mathrm{i}\hbar\,\partial/\partial p, p - \tfrac{1}{2}\mathrm{i}\hbar\partial/\partial x\right)\right] U_t(x,p) = \mathrm{O}(\hbar),$$
$$U_0(x,p) = 1. \tag{35}$$

Consider the phase space $\mathcal{H}^4 = \mathcal{H}^2 \times \mathcal{H}^2 = \{(x,p,q,y)\}$ with the symplectic form $\mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}p \wedge \mathrm{d}q$ generated by the pairing $\langle\cdot,\cdot\rangle$. Consider the Hamiltonian function $\tilde{H}(x,p,q,y) = H\left(x + \tfrac{1}{2}q, p - \tfrac{1}{2}y\right)$. Obviously $\tilde{H} \in \mathcal{D}^2(\mathcal{H}^4, \mathbb{R}^1)$. Eq. (35) can now be written in the form

$$[(\partial/\partial t) - (\mathrm{i}/\hbar)\tilde{H}_t(\xi, \mathrm{i}\hbar\,\partial/\partial\xi)]U_t(\xi) = \mathrm{O}(\hbar),$$
$$U_0(\xi) = 1, \tag{36}$$

where $\xi = (x,p)$.

The solution of such an initial value problem was constructed in [D1,D2] by the method of Maslov's canonical operator. Let $\Lambda_0 = \mathcal{H}^2 \times \{0\} = \{(x_0, p_0, 0, 0)\}$ and $\Lambda_t = \gamma_{\tilde{H}}(0, t)(\Lambda_0)$ Let $\mathcal{K}_{\Lambda_t} : C^\infty(\Lambda_t) \to C^\infty(\mathcal{H}^2)$ be the Maslov canonical operator corresponding to the manifold Λ_t. Then

$$U_t(\xi) = \mathrm{e}^{\mathrm{i}c(t)/\hbar}\mathcal{K}_{\Lambda_t}(1)(\xi) \tag{37}$$

for some $c(t) \in \mathbb{R}^1$. It follows from the results of [D1,D2] that $U_t(x, p) = \mathrm{e}^{\mathrm{i}P_t(x,p)/\hbar} + S_{\hbar,t}(x, p)$, $P \in \mathcal{P}^2$, $S \in \mathcal{F}$. Propositions 3 and 4 imply that $\hat{U}$ is bounded operator (the determinant in formula (22) is not equal to zero because of the structure of the manifold Λ_t). □

Now let us specify the manifold Λ_t. Introduce new variables:

$$x = \tfrac{1}{2}(\alpha + \alpha'), \quad q = \alpha - \alpha', \quad p = \tfrac{1}{2}(\beta + \beta'), \quad y = -\beta + \beta'. \tag{38}$$

Evidently $\tilde{H}(x, q, p, y) = H(\alpha, \beta)$ and

$$\dot{\alpha}(t) = \frac{\partial H}{\partial \beta}, \quad \dot{\beta}(t) = -\frac{\partial H}{\partial \alpha}, \quad \alpha'(t) = x_0, \quad \beta'(t) = p_0. \tag{39}$$

We see that $(\alpha, \beta) = \gamma_t(\alpha', \beta')$ and

$$\Lambda_t = G(\gamma_t) = \{(\alpha', \beta', \alpha, \beta) \colon (\alpha, \beta) = \gamma_t(\alpha', \beta')\} \tag{40}$$

is the graph of γ_t in coordinates (38).

Set for any $\gamma \in \mathrm{Symp}_{\mathcal{F}}$ $U_\gamma = \mathcal{K}_{G(\gamma)}(1)$. As above, $U_\gamma(x, p) = \mathrm{e}^{\mathrm{i}P(x,p)/\hbar} + S_\hbar(x, p)$, $P \in \mathcal{P}^2$, $S \in \mathcal{F}$, and the operator $T_\gamma = \hat{U}_\gamma$ is bounded.

Theorem 4.

(1) *The mapping $\gamma \mapsto T_\gamma$ is an asymptotic projective representation of the group* $\mathrm{Symp}_{\mathcal{F}}$, *i.e.*

$$T_{\gamma_1}T_{\gamma_2} = c_{\gamma_1,\gamma_2}T_{\gamma_1,\gamma_2} + \mathrm{O}(\hbar), \tag{41}$$

where c_{γ_1,γ_2} is some 2-cocycle.

(2) *For $G \in \mathcal{E}$*

$$\hat{G}T_\gamma = T_\gamma(\widehat{\gamma^* G}) + \mathrm{O}(\hbar), \tag{42}$$

where $\gamma^* G = G(\gamma(\cdot))$.

Proof. Notice that for any $\gamma \in \mathrm{Symp}_{\mathcal{F}}$ there exists a Hamilton function $H(t) \in \mathcal{D}^2$ such that $\gamma = \gamma_H(0, 1)$ (Proposition 5). Therefore $T_\gamma = cT_1$, where T_t satisfies Schrödinger equation (33). This implies the statement of the theorem, in a similar way as in the finite-dimensional case [KM1,KM2]. □

Remark 3.

(1) The considerations above give the main terms of the corresponding $\hbar$-expansions. The estimates of the remainders can be obtained by a stationary phase method.

(2) The kernel U_γ is universal in the sense that it does not depend on the choice of the measure η and cocycle α.

(3) In the finite-dimensional case the operators $r^{-1}Tr$ (see formula (18)) coincide mod $O(\hbar)$ with the operators considered in [KM1,KM2].

4. Symplectic manifolds connected with Hamiltonian systems in Hilbert phase space and their asymptotic quantization

Let us now consider a manifold Σ modeled on $\mathcal{H}^2$ with an atlas $\mathcal{U} = \{(U_\xi, \varphi_\xi)\}_{\xi\in\mathbb{A}}$. Let $\psi_{\xi\zeta} = \varphi_\xi\varphi_\zeta^{-1}, \xi, \zeta \in \mathbb{A}$ be the corresponding transition mappings.

Definition 4. The atlas $\mathcal{U}$ will be called an $\mathcal{F}$-atlas, if for the transition mappings ψ we have $\psi_{\xi\zeta} \in \mathrm{Symp}_{\mathcal{F}}(\mathcal{H}^2)$. The set of all equivalent $\mathcal{F}$-atlases on Σ will be called a symplectic $\mathcal{F}$-structure. The manifold Σ with a fixed $\mathcal{F}$-structure will be called a symplectic $\mathcal{F}$-manifold.

Properties 2.

(1) Σ is equipped with the natural symplectic form generated by the forms $\varphi_\xi^*(\mathrm{d}p \wedge \mathrm{d}x)$, $\xi \in \mathbb{A}$.

(2) Σ is a real-analytic manifold.

Example 1. Consider the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{43}$$

where $H \in \mathcal{D}^2$ and is allowed to depend on t as in Theorem 3. Let $\mathcal{M}$ be the space of trajectories $[0, T] \ni t \mapsto \eta(t) \in \mathcal{H}^2$ of the corresponding flow. Consider the mapping φ_t : $\mathcal{M} \ni \eta \mapsto \eta(t) \in \mathcal{H}^2$. From Theorem 3 we immediately have the following proposition.

Proposition 6. *$\mathcal{M}$ is a symplectic $\mathcal{F}$-manifold globally symplectomorphic to $\mathcal{H}^2$. Any atlas formed by the mappings φ_t is an $\mathcal{F}$-atlas on $\mathcal{M}$.* □

Example 2. The next example is given by the manifold of integral curves of the Hamiltonian vector field (43) on a given level of fixed energy. Let $H \in \mathcal{D}^1$ and be independent of t. Suppose that a is a regular value for H, in the sense that $\mathrm{d}H \neq 0$ on the level surface $\mathcal{R} = \{\xi \in \mathcal{H}_-^2 : H(\xi) = a\}$. Consider the space $\mathcal{N} = \mathcal{R}/\gamma_H$ of integral curves of the Hamiltonian system (43) on $\mathcal{R}$.

Let $\mathcal{K} \subset \mathcal{H}$ be a Hilbert subspace of codimension 1 and $(\mathcal{K}^2, \mathrm{d}p \wedge \mathrm{d}x)$ be the corresponding phase space. Then we have the following proposition.

Proposition 7. *$\mathcal{N}$ has the structure of a symplectic $\mathcal{F}$-manifold with the space $(\mathcal{K}^2, \mathrm{d}p \wedge \mathrm{d}x)$ as a model.*

Proof. Consider at first the space $\mathcal{R}$. Let us introduce a manifold structure on $\mathcal{R}$ in the following way. Choose an orthonormal basis in $\mathcal{H}^2$ formed by an orthonormal basis in $\mathcal{H}$ and the corresponding biorthogonal one in $\mathcal{H}'$. Consider an atlas on $\mathcal{R}$ formed by projections on the corresponding coordinate linear spaces of codimension 1. The transition mappings in this case have the form $\mathcal{H}^2 \ominus \mathbb{R}^1 \ni y = (y_1, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots) \mapsto (y_1, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots)$, where the function $\varphi : \mathcal{H}^2 \ominus \mathcal{R}^1 \to \mathcal{R}^1$ locally satisfies the equation $H(y, \varphi(y)) = a$ and is defined by the implicit function theorem. In a similar way as in Theorem 2 it can be shown that $\varphi \in \mathcal{D}^1(H^2_- \ominus \mathcal{R}^1, \mathcal{R}^1)$ and $\psi - \mathrm{id} \in \mathcal{B}(\mathcal{H}^2_- \ominus \mathcal{R}^1)$ (the equation above can be considered as an equation in the space $\mathcal{B}_\lambda$.

The Hamiltonian vector field (43) is tangent to $\mathcal{R}$. Therefore Theorem 2 implies that the factor-space $\mathcal{N} = \mathcal{R}/\gamma_H$ also has a manifold structure with the atlas formed by projections on the corresponding coordinate subspaces of codimension 2 and the transition mappings $\gamma = \mathrm{id} + \theta$, $\theta \in \mathcal{B}(\mathcal{H}^2 \ominus \mathcal{R}^2)$. By the usual construction of Hamiltonian reduction the restrictions of the form $\mathrm{d}p \wedge \mathrm{d}x$ on these subspaces are nondegenerate and form a symplectic structure on $\mathcal{N}$. Therefore these subspaces are symplectomorphic to $\mathcal{K}^2$ and the transition mappings belong to the class $\mathrm{Symp}_{\mathcal{F}}(\mathcal{K}^2)$. □

Our next goal is to construct an algebra of PDO with symbols on a symplectic $\mathcal{F}$-manifold Σ. We consider only the case of simply connected Σ, for example $\Sigma = M$ or Σ is a simply connected domain of $\mathcal{N}$.

Define the space $\mathcal{E}(\Sigma)$ of complex functions Φ on Σ such that for any map (U_ξ, φ_ξ) the function $\Phi\varphi_\xi^{-1} : \varphi_\xi(U_\xi) \to \mathbb{C}^1$ is the restriction on $\varphi_\xi(U_\xi)$ of some function $\Phi_\xi \in \mathcal{E}(\mathcal{H}^2)$. The definition is correct: because of Theorem 1 the mappings of the class $\mathcal{B}(\mathcal{H}^2)$ preserve the class $\mathcal{E}$. Notice that for fixed $\xi \in \mathbb{A}$ the correspondence $\Phi \to \Phi_\xi$ is unique because of the analyticity of Φ_ξ. The space $\mathcal{E}(\Sigma)$ has the structure of a Poisson algebra with the Poisson bracket $\{\cdot,\cdot\}$ generated by the symplectic structure on Σ.

Let us fix a Gaussian measure η and a Gaussian cocycle α and define for each $\xi \in \mathbb{A}$ the mapping $Q_\xi : \Phi \mapsto \hat{\Phi}_\xi$. Because of the results of Section 2, this mapping gives an asymptotic quantization of the Poisson algebra $\mathcal{E}(\Sigma)$:

$$(\mathrm{i}/\hbar)[\hat{\Phi}_\xi, \hat{\Psi}_\xi] = \widehat{\{\Phi, \Psi\}}_\xi + \mathrm{O}(\hbar^2). \tag{44a}$$

Now let us construct a quantization which is invariant under the choice of ξ. For this we shall give an invariant definition of PDO with $\mathcal{E}(\Sigma)$-symbol. Let (U_ξ, φ_ξ), (U_ζ, φ_ζ) be two intersecting maps and let $\gamma_{\xi,\zeta}$ be the corresponding transition mapping. Obviously $\Phi_\xi = \gamma^*_{\xi,\zeta}\Phi_\zeta$ and, because of (41),

$$T_{\gamma_{\xi,\zeta}}\hat{\Phi}_\xi = \hat{\Phi}_\zeta T_{\gamma_{\xi,\zeta}} + \mathrm{O}(\hbar). \tag{44b}$$

Introduce now the (mod $\hbar$) sheaf $\mathcal{S}(\Sigma)$ of linear spaces over Σ putting into the correspondence to each map U_ξ the space $\Gamma = L_2(\mathcal{H}_-, \eta)$ and defining the homomorphisms $r_{\xi\zeta} : \Gamma_\xi \to \Gamma_\zeta$ for $U_\xi \supset U_\zeta$ with the aid of the operators T: $r_{\xi\zeta} = T_{\gamma_{\xi\zeta}}$. The space of sections ψ of this sheaf will be denoted by $\Gamma(\Sigma)$. ψ satisfies the equality

$$\psi \circ \varphi_\xi^{-1} = T_{\gamma_{\xi,\zeta}}(\psi \circ \varphi\zeta^{-1}) + \mathrm{O}(\hbar) \tag{45}$$

on the intersection $U_\xi \cap U_\zeta$.

Let us define the PDO $\hat{\Phi}$ with the symbol $\Phi \in \mathcal{E}(\Sigma)$ in the space $\Gamma(\Sigma)$ by the formula

$$(\hat{\Phi}\psi)\restriction_{U_\xi} = \hat{\Phi}_\xi(\psi\restriction_{U_\xi}). \tag{46}$$

Formulae (43)–(46) lead easily to the following result.

Theorem 5. *The PDO $\hat{\Phi}$ with symbols $\Phi \in S(\Sigma)$ are correctly defined* mod $\mathrm{O}(\hbar)$. *The commutation formula*

$$(\mathrm{i}/\hbar)[\hat{\Phi}, \hat{G}] = \widehat{\{\Phi, G\}} + \mathrm{O}(\hbar) \tag{47}$$

holds on a dense domain (with the Poisson bracket $\{\cdot,\cdot\}$ generated by the symplectic structure on Σ)

Remark 4.

(1) For our infinite-dimensional construction we have extended the general scheme of asymptotic quantization proposed by Karasev and Maslov [KM1,KM2]. However the sheaf appearing in our construction is simpler because the symbols we are considering are restricted to be analytic.

(2) Our definition of PDO is based on properties of the representation of canonical commutation relations in $L_2(\mathcal{H}_-, \eta)$. It seems to be possible to replace the Gaussian measure η by more general smooth measure μ satisfying some additional conditions.

(3) The Dirichlet operator $\mathcal{H}_\mu$ associated with measure μ (see e.g. [AKR,AR,BK]) can also be considered (at least heuristically) as the function of operators satisfying the canonical commutation relations in $L_2(\mathcal{H}, \mu)$. However H_μ cannot be defined by the formula (17). Nevertheless it seems to be possible to obtain for H_μ formulae of the type (27) and (42) and to include it into our quantization scheme. Moreover the unitary group corresponding to H_μ could be used in the construction of invariant PDO with symbols on the solution manifold of the Hamiltonian system associated with the formal symbol of H_μ. In particular, this class of manifolds includes the solution manifold of the wave equation [Se].

References

[ABHK] S. Albeverio, R. Hoegh-Krohn and Ph. Blanchard, Feynman path integrals and the trace formula for Schrödinger operators, Commun. Math. Phys. 83 (1982) 49–76.

[AB] S. Albeverio and Z. Brzeźniak, Finite dimensional approximations approach to oscillatory integrals and stationary phase in infinite dimensions, J. Funct. Anal. 113 (1993) 177–244.

[AHK1] S. Albeverio and R. Hoegh-Krohn, Mathematical theory of Feynman path integrals, *Lecture Notes in Mathematics* Vol. 523 (Springer, Berlin, 1976).

[AHK2] S. Albeverio and R. Hoegh-Krohn, Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics, I, Inv. Math. 40 (1977) 59–107.

[AHK3] S. Albeverio and R. Hoegh-Krohn, Feynman path integrals and the corresponding method of stationary phase, *Lecture Notes in Physics*, Vol. 106 (Springer, Berlin, 1979).

[AKR] S. Albeverio, Yu. Kondrat'ev and M. Röckner, An approximate criterion of essential self-adjointness of Dirichlet operators, Potential Analysis 1(3) (1992) 307–319; Addendum, 2(2) (1993) 195–199.

[AR] S. Albeverio and M. Röckner, New developments in the theory and applications of Dirichlet forms, in: *Stochastic Processes, Physics and Geometry, Proc. 2 Int. Conf. Ascona–Locarno–Como 1988* (World Scientific, Singapore, 1990).

[AP] L. Andersson and G. Peters, Geometric quantization on Wiener manifolds, in: *Stochastic Analysis and Applications, Proc. 1989 Lisbon Conf.*, eds. A.B. Cruseiro and J.C. Zambrini, Progr. Probab. 26 (1991) 29–51.

[BD] Ja. Belopolskaja and Yu. Dalecky, *Stochastic Equations and Differential Geometry* (Kluwer Academic Publishers, Dordrecht, 1990).

[BK] Yu. Beresanskii and Yu. Kondrat'ev, *Spectral Methods in Infinite Dimensional Analysis* (Kluwer Academic Publishers, Dordrecht, 1992).

[B] F. Berezin, *The Method of Second Quantization* (Academic Press, New York, 1966).

[BV] P. Bleher and M. Vishik, On a one class of pseudo differential operators with infinite number of variables and their applications, Math. USSR Sbornik 15(3) (1971) 443–491.

[D1] A. Daletskii, Maslov canonical operator on Lagrangian manifolds with Hilbert model, Math. Notes 48(6) (1990) 1206–1212.

[D2] A. Daletskii, Semiclassical approximations for a one class of infinite dimensional pseudodifferential equations, in: *Methods of Functional Analysis in Mathematical Physics* (Institute for Mathematics, Kiev, 1990) (in Russian); English translation, Sel. Math. Sov., to appear.

[D3] A. Daletskii, Infinite dimensional Schrödinger equations and a representation of a symplectic transformations group of Hilbert phase space, Funct. Anal. Appl. 26(1) (1992) 74–75.

[D4] A. Daletskii, Representations of canonical commutation relations defined by Gaussian measure and Gaussian cocycle, Random Operators and Stochastic Equations 2(1) (1994) 87–95.

[D5] A. Daletskii, Some questions of functional calculus for representations of infinite dimensional canonical commutation relations, preprint No. 177 (1993).

[DF] Yu. Dalecky and S. Fomin, *Measures and Differential Equations in Infinite Dimensional Space* (Kluwer Academic Publishers, Dordrecht, 1991).

[H] D. Henry, *Geometric Theory of Semilinear Parabolic Equations* (Springer, Berlin, 1981).

[KM1] M. Karasev and V. Maslov, Pseudodifferential operators and the canonical operator in general symplectic manifolds, Math. USSR Izv. 47(5) (1983) 271–305.

[KM2] M. Karasev and V. Maslov, Asymptotic and geometric quantization, Russian Math. Surveys 39(6) (1984) 133–197.

[KN] M. Karasev and V. Nasajkinskii, About a quantization of quickly oscillating symbols, Math. USSR Sbornik 34(6) (1978) 737–764.

[Kh] A. Khrennikov, Infinite-dimensional pseudodifferential operators, Math. USSR Izv. 33(3) (1988) 575–601.

[K] V. Kolokoltsov, Maslov index in infinite-dimensional symplectic geometry, Math. Notes 48(6) (1990) 64–68.

[M] V. Maslov, *Théorie des perturbations et métodes asymptotiques* (Dunod, Paris, 1972).

[MF] V. Maslov and M. Fedoryuk, *Semiclassical Approximation in Quantum Mechanics* (Reidel, Dordrecht, 1981).

[Re] J. Rezende, The method of stationary phase for oscillatory integrals on Hilbert spaces, Commun. Math. Phys. 101 (1985) 187–206.

[Sa] Yu. Samoilenko, *Spectral Theory of Families of Self-adjoint Operators* (Kluwer Academic Publishers, Dordrecht, 1987).

[SU] Yu. Samoilenko and G. Us, Differential operators with constant coefficients in function space of a countable number of variables, Selecta Mathematica Sovietica, 9(4) (1990) 378–401.

[Se] I. Segal, Poincaré-invariant structures in the solution manifold of a nonlinear wave equation, Revista Matematica Iberoamericana 2(1, 2) (1986) 99–104.

[S] O. Smoljanov, Infinite-dimensional pseudodifferential operators and Schrödinger quantization, Soviet Math. Dokl. 25(2) (1982) 404–408.